THE MEASUREMENT OF THE THERMAL CONDUCTIVITY OF GASES AT HIGH TEMPERATURE

Analysis of an Alternating Plus Direct Current Hot Wire Method which Avoids Radiation Error

PROGRESS REPORT

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ABSTRACT

A method is mathematically derived for the measurement of the thermal conductivity of gases which is independent of radiation, and thus particularly suitable at elevated temperatures. The method employs a conventional hot wire cell with its wire heated by a sinusoidal alternating current superimposed upon a direct current. The operating conditions under which the radiation introduces no appreciable error are indicated as well as the inherent limitations of the method. Optimum cell design and operating conditions are suggested.

I. Outline of Proposed Method for Determining Thermal Conductivities of Gases at High Temperature

The method proposed to measure thermal conductivity at high temperatures employs a hot wire cell in which the wire is heated by a variable cyclical current. A square wave of direct current or any alternating current whose wave form is known could be considered. However, a sinusoidal alternating current superimposed upon a constant direct current has been assumed in the following mathematical analysis. This wave pattern was chosen, not only for mathematical convenience, but also because the performance of electronic equipment is best known in terms of sinusoidal alternating current.

The principal reason for the mathematical analysis was to obtain the theoretical calibration equations and to determine the optimum cell design and operating conditions such that the response is sensitive to variations in thermal conductivity in the range expected and insensitive to wire-to-wall radiation. However, since it most likely will prove necessary to use empirical calibration to account for unavoidable errors, any repetitive current shape could in practice be employed.

The method developed in this paper is based upon the fact that the temperature of the wire will oscillate above and below its average temperature out of phase with the input power due to the fact that the gas has a finite thermal conductivity, specific heat, and density and thus does not instantaneously remove the heat generated within the wire. Therefore any physical or electrical property which is a function of this phase lag may in theory be measured to determine the thermal conductivity of the gas. The following mathematical analysis applies this principle and indicates the properties which may be measured.

II. Mathematical Analysis of Differential Equation and Accompanying Boundary Conditions

A wire of radius r_1 mounted coaxially in a cylindrical jacket of radius r_2 is heated with superimposed A.C. and D.C. currents. Heat is removed from the wire both by conduction through the gas which fills the jacket and by radiation from the wire to the wall. In order to find the temperature of the wire as a function of time, the following assumptions are made:

1. The wire is at a uniform temperature, t_1, at any instant (true for small wires and low frequencies).

2. All physical properties are independent of temperature (true for small temperature fluctuations).

3. The external jacket is maintained at a constant uniform temperature, t_2 (true for a heavy jacket at steady state).

4. Heat radiated by the wire is independent of time and proportional to the average power imput to the wire (true for reasonably small net radiation and temperature fluctuations small compared to the wire-to-jacket temperature difference. It is to satisfy this condition that D.C. is required as well as A.C.)

For any desired test conditions the extent of deviation from each of these assumptions can be computed.

The differential equation that applies in the gas phase is, in cylindrical coordinates:

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} = \frac{1}{\alpha}\frac{\partial t}{\partial \theta} \qquad r_1 < r < r_2 \qquad (1)$$

The boundary conditions at the wire and jacket surfaces are obtained by heat balances.

Thus at $r = r_1$

Rate of total heat into wire: $(i + I_o \sin\omega\theta)^2 R$

Rate of heat absorbed by wire: $\pi r_1^2 c'\rho' L \frac{\partial t}{\partial \theta}$

Rate of heat conducted into gas: $-2\pi r_1 kL \frac{\partial t}{\partial r}$

Rate of heat radiated to jacket: $\gamma Q' = \gamma \left(i^2 + \frac{I_o^2}{2}\right)R$

Equating the heat input to heat absorbed plus heat transferred yields:

$$(i + I_o \sin\omega\theta)^2 R = \pi r_1^2 c'\rho' L \frac{\partial t}{\partial \theta} - 2\pi r_1 kL \frac{\partial t}{\partial r} + \gamma\left(i^2 + \frac{I_o^2}{2}\right)R \qquad (2)$$

A similar heat balance at the jacket wall yields:

$$(1 - \gamma)\left(i^2 + \frac{I_o^2}{2}\right)R = -\ 2\pi r_2 kL \frac{\partial t}{\partial r} \qquad (3)$$

The solution of the differential equation, using the previously stated assumptions and boundary values is presented in the Appendix. The following expression is obtained for the temperature of the wire as a function of time:

$$t_1 = t_2 + \frac{(i^2+I^2)(1-\gamma)R}{2\pi kL} \ln \frac{r_2}{r_1} + \frac{I^2 R E_3''}{\pi r_1^2 2\omega c'\rho' L} \cos(2\omega\theta + \emptyset) +$$

$$\frac{2\sqrt{2}\ iIRE_2''}{\pi r_1^2 \omega c'\rho' L} \cos(\omega\theta + \epsilon) \qquad (4)$$

where $\sqrt{2}\ I = I_o$ and E_3'', E_2'', $\emptyset$, ϵ are dimensionless quantities which are complex functions of the geometry of the apparatus and the physical properties of the wire and gas. The relationship between the variables can be expressed in terms of the following or equivalent dimensionless groups:

$$E_3'' = F_1\left[\left(\frac{\omega r_1^2}{\alpha}\right)\left(\frac{c'\rho'}{c\rho}\right)\left(\frac{r_2}{r_1}\right)\right] \quad (5) \qquad \phi = F_3\left[\left(\frac{\omega r_1^2}{\alpha}\right)\left(\frac{c'\rho'}{c\rho}\right)\left(\frac{r_2}{r_1}\right)\right] \quad (6)$$

$$E_2'' = F_2\left[\left(\frac{\omega r_1^2}{\alpha}\right)\left(\frac{c'\rho'}{c\rho}\right)\left(\frac{r_2}{r_1}\right)\right] \quad (7) \qquad \epsilon = F_4\left[\left(\frac{\omega r_1^2}{\alpha}\right)\left(\frac{c'\rho'}{c\rho}\right)\left(\frac{r_2}{r_1}\right)\right] \quad (8)$$

It is important to note that none of these dimensionless ratios is a function of radiation, which only enters in the coefficient of the log term in equation 4. Therefore the measurement of any one of these four quantities, together with the knowledge of the required physical properties and the dimensions of the apparatus, should yield the value of α of the test gas without any corrections for wire to wall radiation, provided the cell is operated so that the previously stated assumptions are valid.

III. Measurement of the Magnitudes and Phase Angles of the Temperature Waves

An electrical method may be derived for the measurement of the above-mentioned dimensionless groups, E_3, E_2, ϕ, ϵ, in the following manner. In the previous derivation it was assumed that the resistance of the platinum wire used in the cell was constant. If it is now assumed that the wire resistance is a linear function of temperature over the range of wire temperature fluctuation, the resistance of the wire can be expressed as $R = R_o(1 + st_1)$, where R_o is the extrapolated value of R at t = 0°C. The voltage drop across the wire becomes

$$E = (i + I_o \sin\omega\theta)R = (i + I_o \sin\omega\theta)R_o(1 + st_1) \quad (9)$$

and combined with equation (4) yields

$$E = (iR_o + \sqrt{2}\, IR_o \sin\omega\theta)\left(1 + st_2 + \frac{(i^2+I^2)(1-\gamma)\bar{R}s}{2\pi kL}\ \ln\frac{r_2}{r_1}\right)$$

$$- \frac{\sqrt{2}\, s\bar{R}\, R_o I^3 E_3''}{2\pi r_1^2 \omega c'\rho' L}\sin(\omega\theta + \phi) - \frac{2iI^2\bar{R}R_o sE_2''}{\pi r_1^2 \omega c'\rho' L}\sin\epsilon$$

$$+ \frac{\sqrt{2}\, s\bar{R}R_o I^3 E_3''}{2\pi r_1^2 \omega c'\rho' L}\sin(3\omega\theta + \phi) + \frac{2iI^2\bar{R}R_o sE_2''}{\pi r_1^2 \omega c'\rho' L}\sin(2\omega\theta + \epsilon) \quad (10)$$

Therefore, the voltage drop across the wire contains a d.c. component, a fundamental frequency component, and second and third harmonics. Consequently

$$E_2'' = \frac{E_2 \pi r_1^2 c' \rho' L \omega}{2iI^2\bar{R}^2}\left(\frac{1}{s} + \bar{t}\right) \tag{11}$$

$$E_3'' = \frac{E_3 \sqrt{2}\pi r_1^2 c' \rho' L \omega}{I^3\bar{R}^2}\left(\frac{1}{s} + \bar{t}\right) \tag{12}$$

where E_2 is the amplitude of the second harmonic voltage component, E_3 is the amplitude of the third harmonic voltage component, and ϵ and ϕ are the phase angles of the second and third harmonic voltage components, respectively. $\bar{t}$ is the average temperature of the wire and $\bar{R}$ is the resistance of the wire at $\bar{t}$.

Therefore, in principle, the measurement of the amplitude of the second or third harmonic voltage components, or their corresponding phase angles, will give values of thermal conductivity independent of wire-to-wall radiation. It can be shown that the phase lag plus the phase angle (ϕ or ϵ) equals π radians.

Initially it was believed permissible to use a pure sinusoidal input with no superimposed direct current. In this case $i = 0$ and the second harmonic component of voltage is absent. For this reason only the equations for E_3'' and ϕ given in Appendix C were evaluated as functions of the dimensionless groups listed in equations (5)-(8). The values were computed to ten significant places using the IBM Electronic Data Processing Machine-Type 701. The significant portion of the results is tabulated in Tables I and II and plotted in Figures 1 through 7. (The tables and the figures are found in the Appendix.) However, it may prove preferable to employ the second harmonic due to lesser build-up in the required electrical circuits. In this case an entirely empirical calibration could be used.

Figure 5 shows that E_3 is most sensitive to changes in α at values of $(\beta r_2)^2$ near 4 at the practical ratio of r_1/r_2 equal to 0.007. At values of $(\beta r_2)^2$ above approximately 2 the initial assumptions are reasonably

valid. It is expected that E_2'' will be less suited for this purpose, although no calculations of E_2'' have been made. The initial A.C. current will undoubtedly have a third harmonic component, which will of course have to be filtered out of the A.C. before it is applied to the hot wire. This third harmonic will be useful, however, as a standard from which to measure changes in ϕ between calibration and test runs.

IV. Analysis of Calculated Data and Graphs

The dimensionless group E_3'' and the phase lag $\Delta = \pi - \phi$ were calculated at various values of βr_2, r_1/r_2, and $c'\rho'/c\rho$ (where $\beta = \sqrt{2\omega/\alpha}$) and plotted as the logarithm of either $\left[(E_3'')_{max} - E_3''\right]$ or $(\Delta_{max} - \Delta)$ versus βr_2 at constant values of r_1/r_2 and $c'\rho'/c\rho$. In addition advantage was taken of the fact that both E_3'' and Δ, as well as their slopes, are zero at βr_2 equal to zero. The curves in the range in which βr_1 is less than 0.01 is dotted in order to indicate their approximate shape, since no values of the required Bessel Functions were available in this region. In any event it would not be desirable to use such low values, because it would mean an extremely small wire radius and/or a very low frequency, and consequently a large wire temperature fluctuation would result.

The following conclusions have been reached by consideration of the data and accompanying graphs:

1. As βr_2 increases, E_3'' approaches 1/2 as a limit and Δ approaches the value of $\pi/2$ radians (90°). This is also the case as $c'\rho'/c\rho$ approaches ∞.

2. For elevated temperatures and normal or low pressures (high values of $c'\rho'/c\rho$), E_3'' varies sufficiently with βr_2 to be suitable for measuring k. In particular, this response may be used for gases at high pressures and moderate temperatures.

3. It is expected for elevated temperatures that the phase lag will be sufficiently sensitive in the range of βr_2 between 1.5 and 3, and r_1/r_2 equal to

approximately 0.007, to measure the thermal conductivity within one or two per cent. This, however, would mean that for most common gases the fundamental frequency must be fairly small. It has been estimated that 60 cycle alternating current should be useful for air if interference from power lines is not objectionable.

V. Conditions for Evaluating Optimum Ratio of Direct Current to Superimposed Alternating Current

From the practical viewpoint of experimental design and operation, the most convenient method would be the limiting case of a pure sinusoidal current with no superimposed direct current. However, it has been calculated for this case that at optimum operating conditions the temperature fluctuation of the wire is about 30 to 80 per cent of the average temperature difference between the wire and the outer jacket. This would mean that the assumption that instantaneous radiation in any run was constant and proportional to the average power input to the wire was not valid and the **previously** mentioned dimensionless groups would have to be corrected for radiation effects.

In order to decrease the temperature fluctuation of the wire to a small fraction of the average temperature difference between the wire and outer wall, direct current is superimposed on the alternating current. The restrictions on the ratio of direct current to alternating current are as follows:

1. The third harmonic component of voltage must be an accurately measureable quantity. Thus there is a minimum alternating current input.

2. Because it is assumed that all physical properties are constant, the temperature difference between the wire and outer jacket must be as small as possible. Therefore, the total power input to the wire must be kept small by limiting the amount of direct current to the wire after the A.C. is set at its operable minimum.

It will not be possible to satisfy the conditions of a small temperature difference and a high ratio of direct current to alternating current simultaneously if a minimum value of alternating current

is maintained. Therefore it is necessary to determine the ratio which gives the most reasonable approach to satisfying these restrictions. It is expected that this optimum ratio of direct current to the effective alternating current will be approximately five with a maximum average temperature different of about 40°C and a minimum effective voltage of the third harmonic in the range of 10^{-6} to 10^{-5} volts.

VI. Evaluation of the Radiation Assumption

Due to the importance of the assumption that radiation is constant in any run and proportional to the average power input in obtaining the result that E_3'', E_2'', ϕ, and ϵ are independent of wire-to-wall radiation, it is desirable to check whether this assumption is valid. A more rigorous solution of this boundary value problem would be obtained by recognizing that radiation is temperature dependent according to the Stefan-Boltzman law; thus

$$q_r = \sigma F_A F_E (2\pi r_1 L)(T_1^4 - T_2^4) =$$
$$\sigma F_A F_E T_2^3 \left[1 + \frac{T_1}{T_2} + \left(\frac{T_1}{T_2}\right)^2 + \left(\frac{T_1}{T_2}\right)^3\right](2\pi r_1 L)(t_1 - t_2) \quad (13)$$

It can be further shown that if the temperature fluctuation of the wire is 2% of the average temperature difference between the wire and jacket, the assumption that $\sigma F_A F_E T_2^3 \left[1 + \left(\frac{T_1}{T_2}\right) + \left(\frac{T_1}{T_2}\right)^2 + \left(\frac{T_1}{T_2}\right)^3\right]$ is a constant and equal to a heat transfer coefficient, h, will result in a maximum error in the value of q_r of about 0.15%. The heat balance on the wire as previously stated

$$M\frac{\partial t}{\partial r} - N\frac{\partial t}{\partial \theta} = (i + I_o \sin\omega\theta)^2 R - \gamma Q' \quad (2)$$

can be modified for this more rigorous treatment to the

form

$$M \frac{\partial t}{\partial r} - N \frac{\partial t}{\partial \theta} = (1 + I_o \sin \omega\theta)^2 R - \gamma Q' \left(\frac{t_1 - t_2}{\bar{t}_1 - t_2} \right) \quad (14)$$

where $(q_r)_{avg} = \gamma Q' = hN(\bar{t}_1 - t_2)$ and $q_r = hN(t_1 - t_2)$.

If equation 14 is correct, then the maximum error in the left-hand side of the original heat balance, equation 2, will be less than 0.05%, providing that the wire temperature fluctuation is about 2% and that the fraction of the total power input transferred by radiation (γ) is less than 1/5. However, because this error is absorbed by every term involving r and θ, then it is expected that the error associated with either the phase lag, Δ, or the amplitude of third harmonic, E_3, is negligible.

In any event, if it does become necessary to make some correction for radiation, and since h and the true average r_1 may not be accurately determinable, an empirical correction for the given apparatus would probably be more satisfactory than a theoretical one. From the mathematical analysis, it can be shown that the radiation correction will be a function of hr_1/k in addition to the previously mentioned dimensionless groups. Substituting in the definition of the heat transfer coefficient in terms of the variables combined in the Stefan-Boltzman relationship which are equivalent to it, yields the following relationship for the radiation correction:

$$\text{Radiation Correction} = F\left[(\beta r_2)\left(\frac{r_1}{r_2}\right)\left(\frac{c'\rho'}{c\rho}\right)\left(\frac{T_1}{T_2}\right)\left(\frac{\sigma F_E F_A r_1 T_2^3}{k}\right)\right] \quad (15)$$

VII. Heat Capacity of the Gas

Throughout the derivation the heat capacity of the gas was used, without indicating whether it was the heat capacity at constant pressure (c_p) or at constant volume (c_v). It has not been definitely determined which term is the correct one to use. Thus it is planned to ascertain this by using blank runs at low temperature with gases of known thermal conductivity.

However, if the temperature fluctuation is considered to move in the gas radially in the form of temperature waves, then flow waves will be correspondingly moving through the gas in the same manner. Since these fluctuations are occuring at about 60 times per second, the flow waves must be very close together, and the process may be considered to be occurring at constant pressure. Therefore it is expected that c_p will be the proper value to use.

VIII. Conclusions

1. It is shown that it is theoretically possible to obtain the thermal conductivity of gases at high temperature from electrical responses which are sufficiently sensitive to thermal conductivity but are insensitive to wire-to-wall radiation. A hot wire cell would be employed in which the wire is heated by a sinusoidal alternating current superimposed upon a direct current.

2. The phase lag of the third harmonic of voltage produced across the wire with respect to some standard, such as the third harmonic present in the unfiltered generator output, appears to be a sensitive indicator of thermal diffusivity for gases at high temperature.

3. It also seems feasible to use the amplitude of the third harmonic to measure the thermal diffusivity of gases.

4. It seems possible to use the amplitude of the third harmonic to measure the thermal diffusivity of liquids. However, calculations at the low volumetric heat capacity ratios ($c'\rho'/c\rho$) that would be obtained with liquids have not yet been carried out.

ACKNOWLEDGMENT

The assistance of Dr. Fritz Steinhardt, of the Department of Mathematics, who checked the derivation of the equations for pure sinusoidal current without D.C. and computed several trial points, is gratefully recognized.

APPENDIX A

Table I. Computed Values of Phase Lag Δ of Third Harmonic Voltage E_3 behind the A.C. Current

r_1/r_2	βr_2	βr_1	Δ at various values of $c'\rho'/c\rho$: $c'\rho'$ = 1000	2000	3000	4000	6000	9000	12000	15000	25000	40000
0.002	10.00	0.020	0.8002	1.0789	1.2164	1.2954	1.3812	1.4420	1.4733	1.4924	1.5234	1.5411
	8.00	0.016	0.6037	0.8754	1.0409	1.1469	1.2709	1.3637	1.4130	1.4435	1.4934	1.5221
	6.00	0.012	0.4519	0.6747	0.8395	0.9602	1.1185	1.2492	1.3225	1.3690	1.4470	1.4925
0.004	10.00	0.040	1.2537	1.4018	1.4561	1.4840	1.5125	1.5317	1.5414	1.5427	1.5566	1.5619
	8.00	0.032	1.1275	1.3255	1.4024	1.4428	1.4844	1.5128	1.5271	1.5358	1.5497	1.5576
	6.00	0.024	0.9212	1.1805	1.2959	1.3594	1.4267	1.4734	1.4973	1.5117	1.5352	1.5485
	4.00	0.016	0.6031	0.8754	1.0412	1.1472	1.2712	1.3640	1.4133	1.4437	1.4936	1.5220
0.005	10.00	0.050	1.3465	1.4539	1.4919	1.5113	1.5309	1.5441	1.5507	1.5547	1.5611	1.5648
	8.00	0.040	1.2534	1.4018	1.4561	1.4840	1.5125	1.5317	1.5414	1.5472	1.5566	1.5619
	6.00	0.030	1.0919	1.3015	1.3850	1.4292	1.4750	1.5064	1.5223	1.5319	1.5474	1.5561
	4.00	0.020	0.8001	1.0792	1.2167	1.2957	1.3815	1.4422	1.4735	1.4926	1.5235	1.5411
	2.00	0.010	0.3137	0.5258	0.6965	0.8295	1.0134	1.1718	1.2625	1.3203	1.4174	1.4741
0.007	10.00	0.070	1.4401	1.5040	1.5260	1.5370	1.5482	1.5557	1.5595	1.5617	1.5654	1.5674
	8.00	0.056	1.3823	1.4733	1.5052	1.5213	1.5377	1.5486	1.5542	1.5575	1.5628	1.5658
	6.00	0.042	1.2750	1.4138	1.4644	1.4903	1.5168	1.5350	1.5436	1.5490	1.5577	1.5626
	4.00	0.028	1.0361	1.2651	1.3590	1.4091	1.4612	1.4970	1.5152	1.5262	1.5439	1.5540
	2.00	0.014	0.4950	0.7051	0.8603	0.9745	1.1256	1.2520	1.3236	1.3694	1.4466	1.4921
0.010	10.00	0.100	1.4976	1.5338	1.5460	1.5522	1.5583	1.5625	1.5646	1.5658	1.5678	1.5689
	8.00	0.080	1.4656	1.5173	1.5349	1.5438	1.5528	1.5587	1.5618	1.5636	1.5664	1.5681
	6.00	0.060	1.4031	1.4845	1.5127	1.5271	1.5415	1.5512	1.5561	1.5590	1.5637	1.5664
	4.00	0.040	1.2540	1.4021	1.4563	1.4842	1.5126	1.5318	1.5415	1.5473	1.5567	1.5620
	2.00	0.020	0.7823	1.0599	1.2003	1.2819	1.3713	1.4350	1.4679	1.4881	1.5207	1.5394

0.003 0.010

APPENDIX A

Table II. Computed Values of E_3'' (Dimensionless Amplitude of the Third Harmonic of the Voltage across the Wire)

r_1/r_2	βr_2	βr_1	$c'\rho' = 1000$	2000	3000	4000	6000	9000	12000	15000	25000	40000
			E_3'' at various values of $c'\rho'/c\rho$									
0.002	10.00	.020	0.2912	0.3949	0.4353	0.4548	0.4728	0.4834	0.4882	0.4909	0.4948	0.4969
	8.00	.016	0.2094	0.3260	0.3857	0.4186	0.4510	0.4707	0.4796	0.4845	0.4915	0.4950
	6.00	.012	0.1415	0.2456	0.3151	0.3607	0.4124	0.4474	0.4637	0.4727	0.4856	0.4918
0.004	10.00	0.040	0.4389	0.4734	0.4834	0.4881	0.4924	0.4951	0.4964	0.4972	0.4983	0.4990
	8.00	0.032	0.4050	0.4586	0.4747	0.4820	0.4887	0.4928	0.4947	0.4959	0.4976	0.4985
	6.00	0.024	0.3371	0.4240	0.4538	0.4677	0.4803	0.4878	0.4912	0.4932	0.4961	0.4976
	4.00	0.016	0.2099	0.3266	0.3863	0.4190	0.4513	0.4710	0.4798	0.4846	0.4916	0.4951
0.005	10.00	0.050	0.4600	0.4822	0.4887	0.4918	0.4947	0.4965	0.4974	0.4980	0.4988	0.4992
	8.00	0.040	0.4389	0.4734	0.4834	0.4881	0.4924	0.4951	0.4964	0.4971	0.4983	0.4990
	6.00	0.030	0.3909	0.4514	0.4702	0.4788	0.4867	0.4916	0.4938	0.4952	0.4972	0.4983
	4.00	0.020	0.2918	0.3954	0.4357	0.4551	0.4730	0.4836	0.4883	0.4910	0.4949	0.4969
	2.00	0.010	0.1217	0.2214	0.2945	0.3457	0.4062	0.4474	0.4660	0.4759	0.4889	0.4944
0.007	10.00	0.070	0.4780	0.4898	0.4934	0.4951	0.4968	0.4979	0.4984	0.4987	0.4993	0.4995
	8.00	0.056	0.4673	0.4852	0.4906	0.4931	0.4955	0.4971	0.4978	0.4983	0.4990	0.4994
	6.00	0.042	0.4440	0.4755	0.4847	0.4889	0.4929	0.4954	0.4966	0.4973	0.4984	0.4990
	4.00	0.028	0.3772	0.4455	0.4670	0.4767	0.4856	0.4910	0.4935	0.4949	0.4970	0.4982
	2.00	0.014	0.1369	0.2370	0.3045	0.3496	0.4021	0.4390	0.4569	0.4670	0.4820	0.4895
0.010	10.00	0.100	0.4875	0.4940	0.4961	0.4971	0.4981	0.4987	0.4990	0.4992	0.4995	0.4997
	8.00	0.080	0.4823	0.4917	0.4946	0.4960	0.4974	0.4983	0.4987	0.4990	0.4994	0.4996
	6.00	0.060	0.4713	0.4869	0.4916	0.4938	0.4960	0.4974	0.4980	0.4984	0.4991	0.4994
	4.00	0.040	0.4394	0.4737	0.4836	0.4882	0.4925	0.4952	0.4964	0.4972	0.4983	0.4990
	2.00	0.020	0.2803	0.3864	0.4293	0.4503	0.4699	0.4816	0.4869	0.4899	0.4943	0.4966

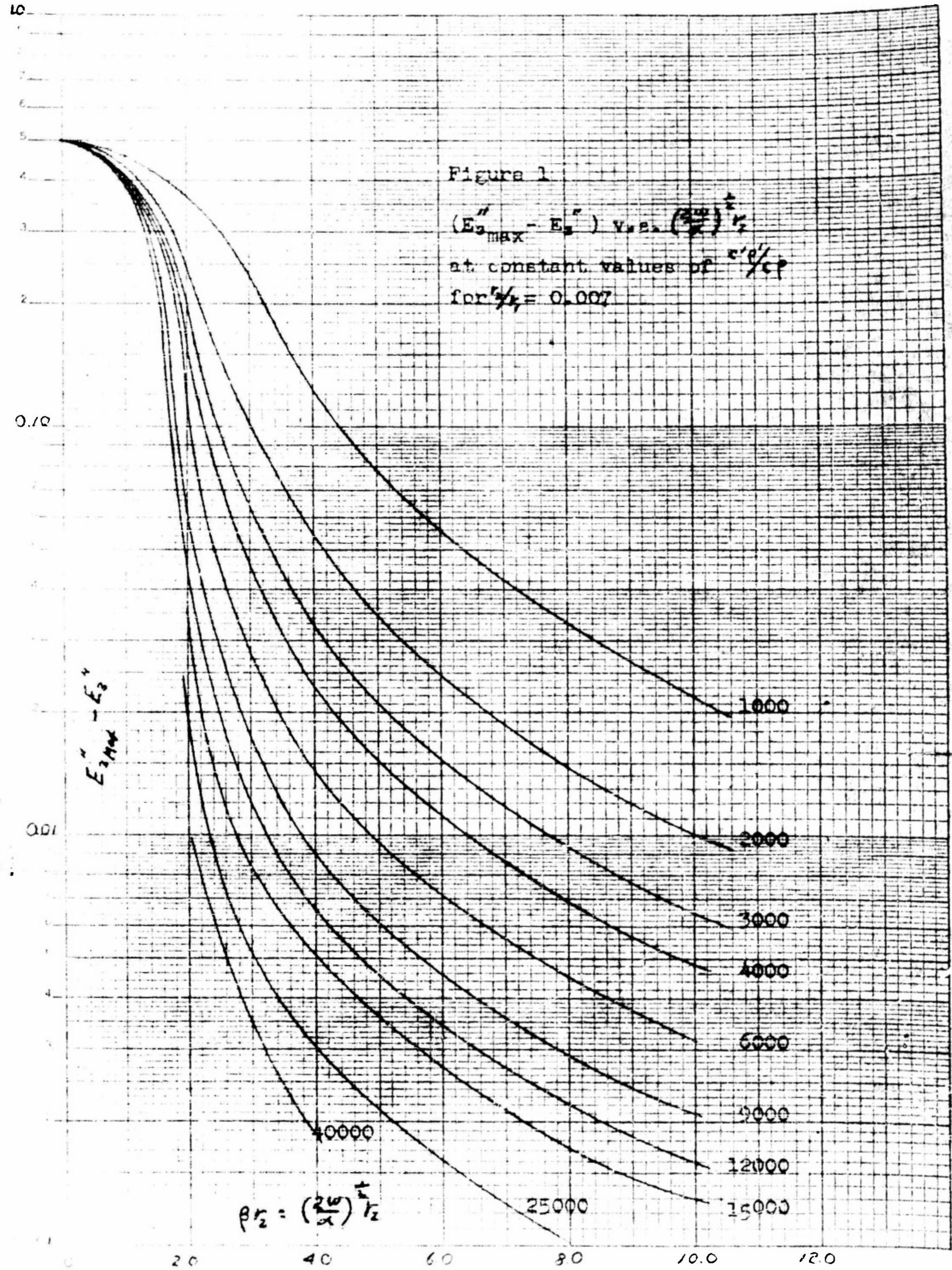


Figure 1
$(E''_{2_{max}} - E''_2)$ vs. $(\frac{2\omega}{\alpha})^{\frac{1}{2}} r_2$
at constant values of $r'\rho'/c\rho$
for r_3/r_2 = 0.007
1000
2000
3000
4000
6000
9000
12000
15000
25000
40000
$E''_{2_{Max}} - E''_2$
$\beta r_2 = (\frac{2\omega}{\alpha})^{\frac{1}{2}} r_2$
0.10
0.01
2.0
4.0
6.0
8.0
10.0
12.0

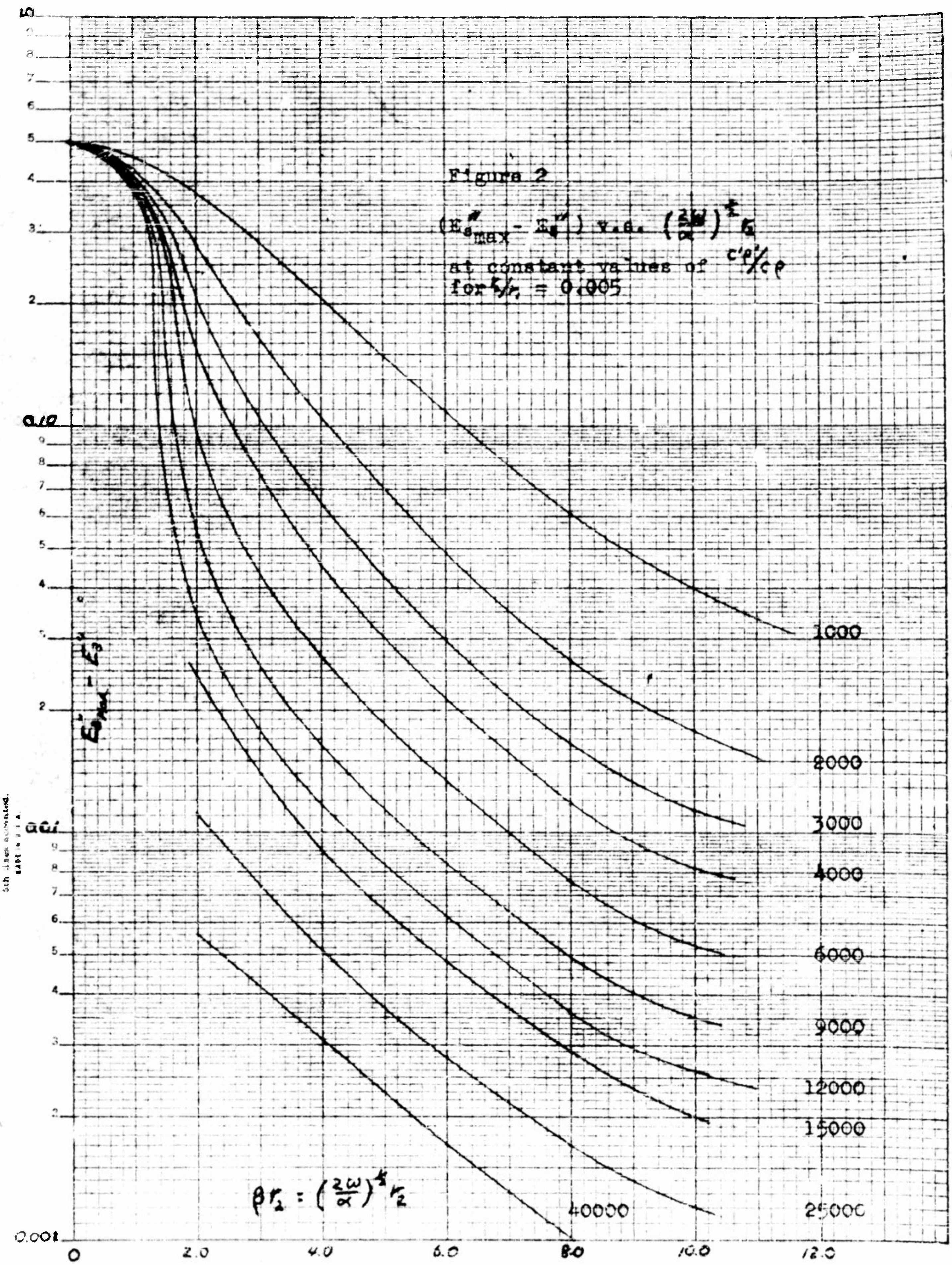


Figure 2

$(E''_{s_{max}} - E''_s)$ v.s. $\left(\frac{2\omega}{\alpha}\right)^{\frac{1}{2}} r_2$

at constant values of $c'\rho'/c\rho$

for $\delta/r_1 = 0.005$

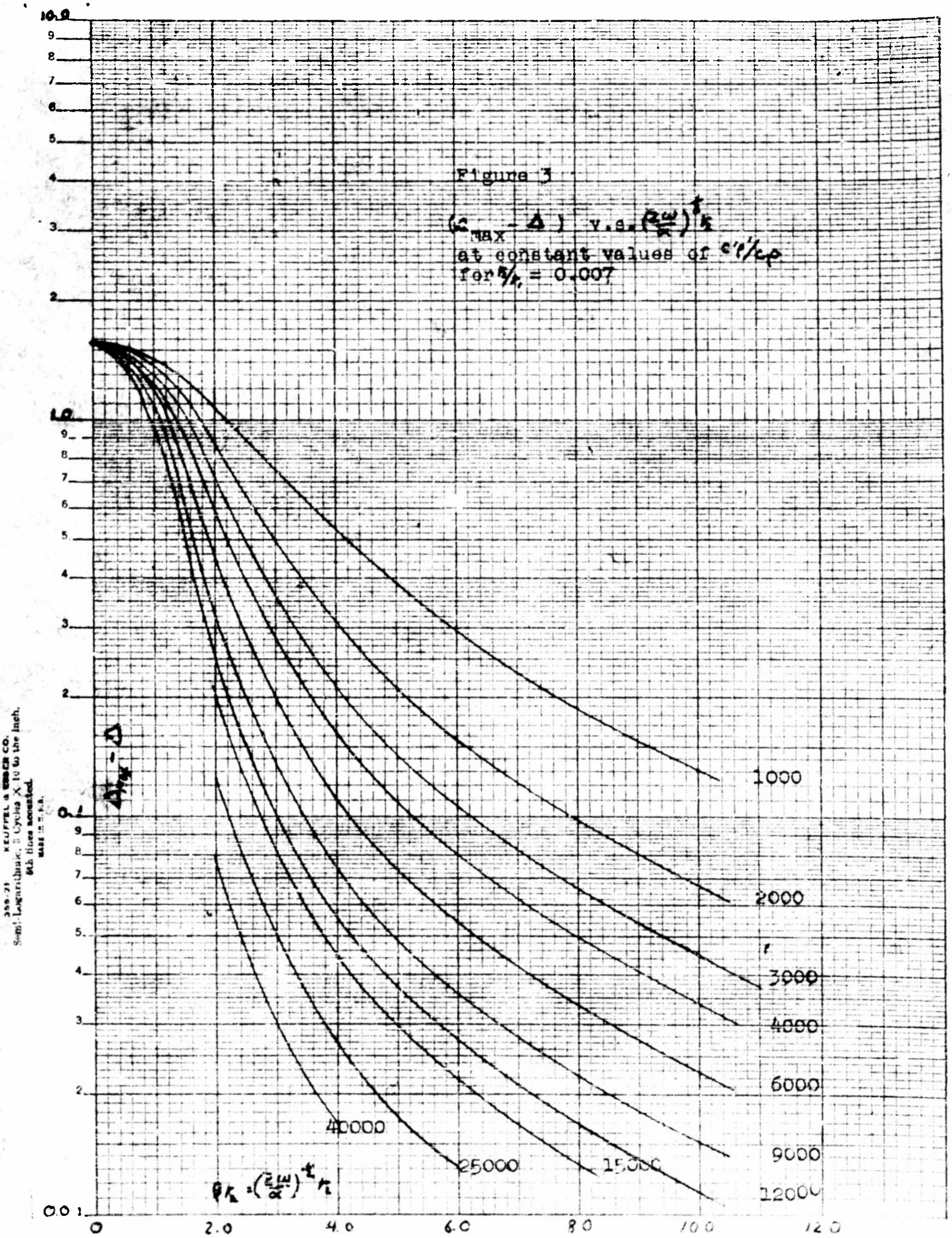


Figure 3

$(\Delta_{max} - \Delta)$ v.s. $\left(\frac{2\omega}{\alpha}\right)^{\frac{1}{2}} r_2$

at constant values of $c'\rho'/c\rho$

for $\delta/r_1 = 0.007$

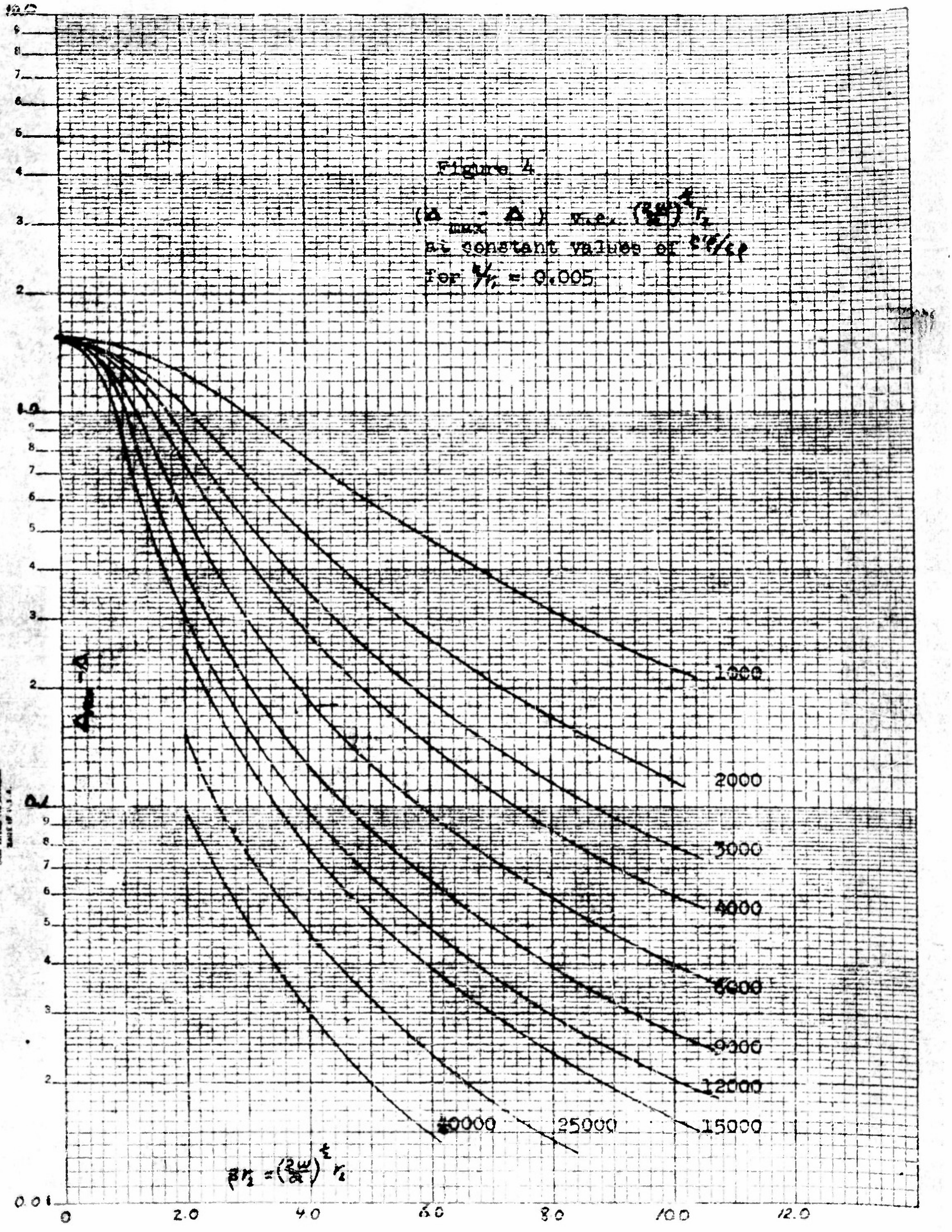


Figure 4
at constant values of
For = 0.005
1000
2000
3000
4000
6000
9000
12000
15000
25000
40000
0
2.0
4.0
6.0
8.0
10.0
12.0
10.0
1.0
0.1
0.01

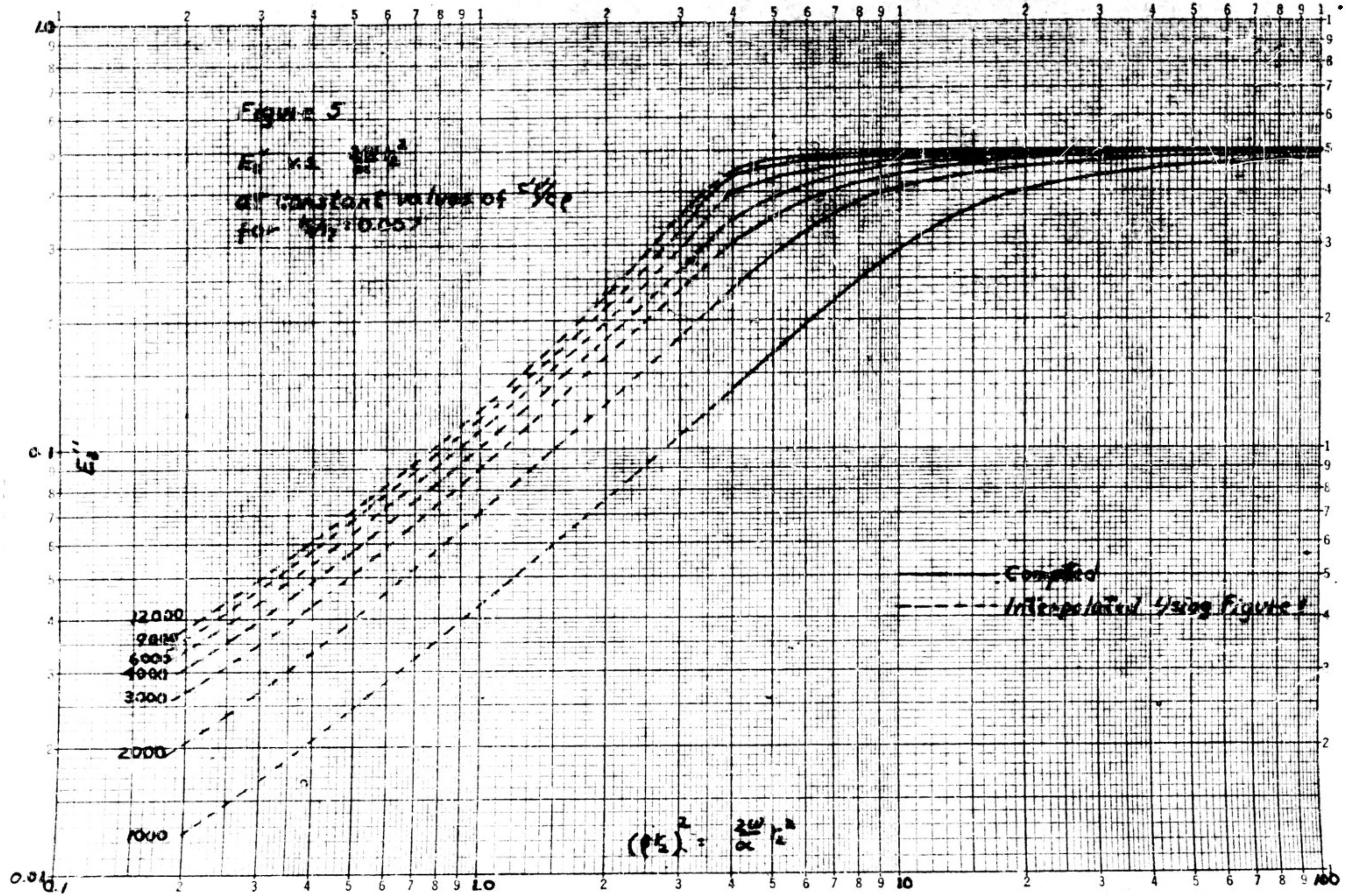


Figure 5
Computed
Interpolated Using Figure 4
12000
6000
4000
3000
2000
1000
0.01
0.1
1.0
10
100

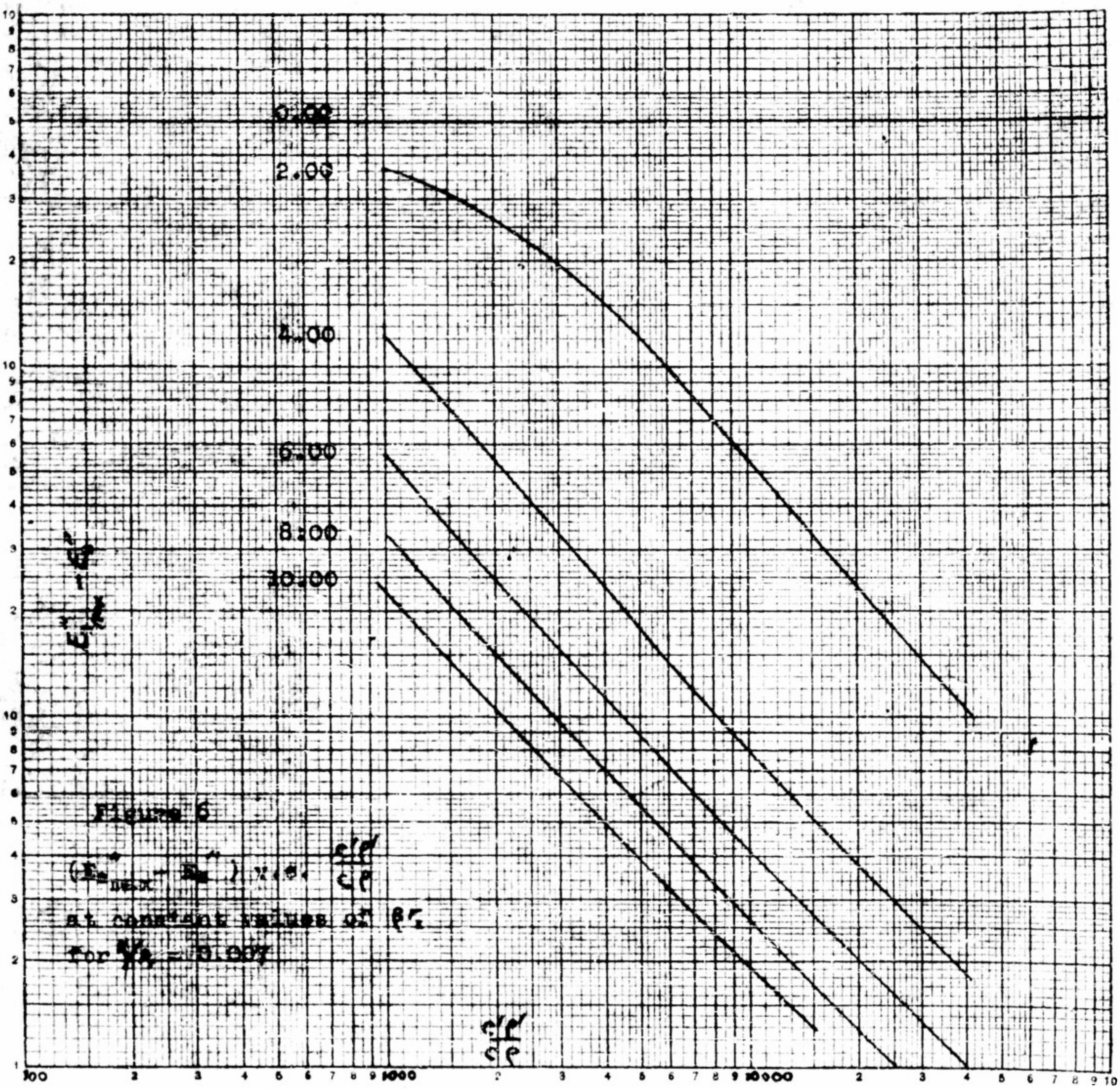
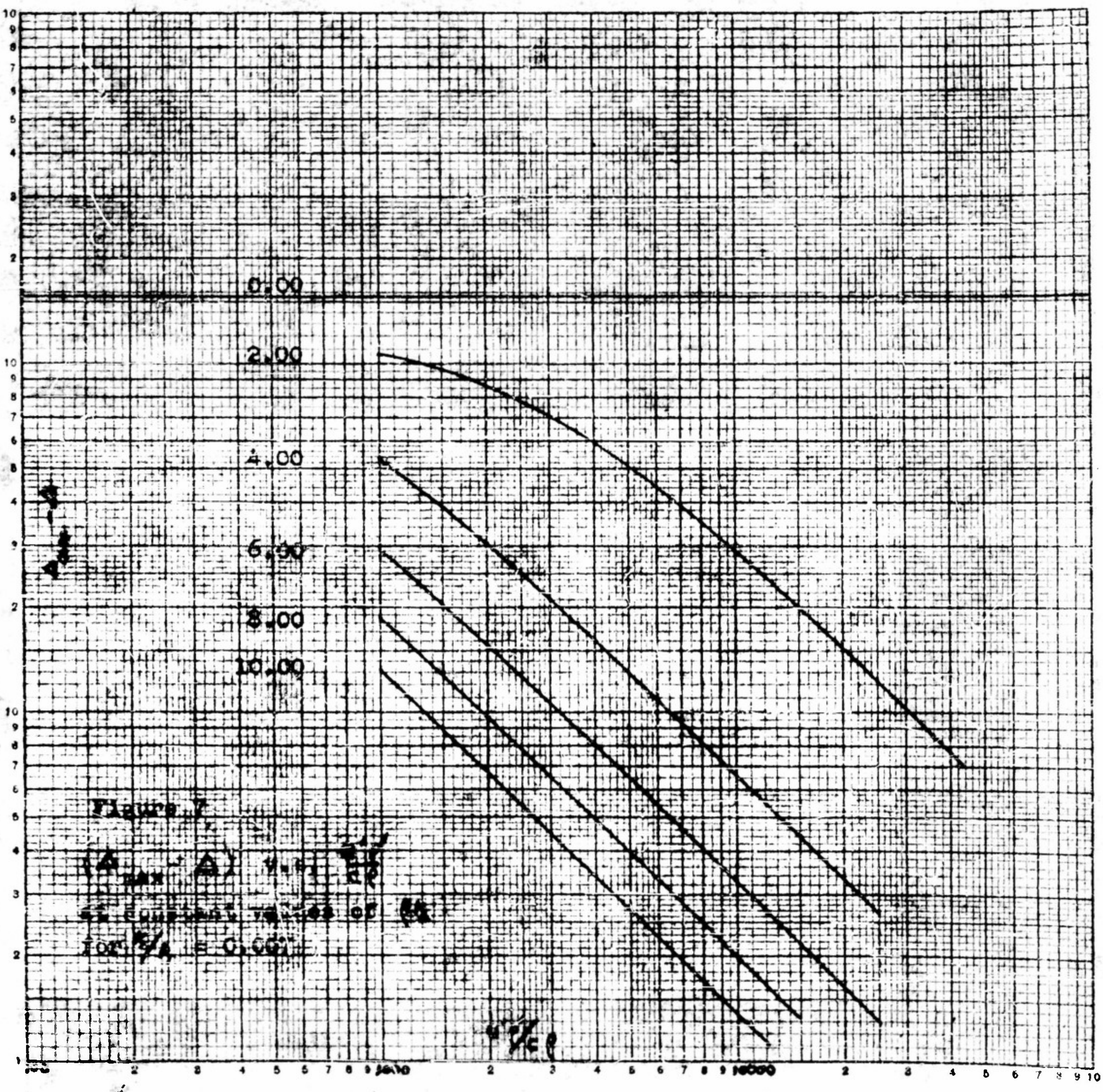


Figure 6

$(E^*_{max} - E^*_s)$ vs. $\frac{c'\rho'}{c\rho}$

at constant values of βr_1

for $y/r_1 = 0.007$


0.00
2.00
4.00
6.00
8.00
10.00
Figure 7.
at constant values of
for

APPENDIX B

Solution of the Differential Equation

The differential equation and the boundary conditions may be written as follows:

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} = \frac{1}{\alpha}\frac{\partial t}{\partial \theta} \quad \text{where } r_1 < r < r_2 \tag{1A}$$

$$(1 + I_o \sin\omega\theta)^2 R - \gamma Q' = M\frac{\partial t}{\partial \theta} - kN\frac{\partial t}{\partial r} \quad \text{at } r = r_1 \tag{2A}$$

$$\frac{\partial t}{\partial r} = \frac{-Q'(1-\gamma)}{kN}\frac{r_1}{r_2} \quad \text{at } r = r_2 \quad \text{where:} \tag{3A}$$

$$M = \pi r_1^{\,2} c'\rho' L$$

$$N = 2\pi r_1 L \quad \text{and}$$

$$Q' = \frac{1}{2\pi}\int_0^{2\pi}(1 + I_o \sin^2\omega\theta)R\,d(\omega\theta) = \left(1^2 + \frac{I_o^{\,2}}{2}\right)R$$

The solution of this boundary value problem is obtained by assuming a solution in the form

$$t = f + g\cos 2\omega\theta + h\sin 2\omega\theta + m\sin\omega\theta + \ell\cos\omega\theta \tag{4A}$$

where f, g, h, l, m are unknown functions of r and independent of θ.

Combining equations 1A through 4A yields the following set of equations

$$f'' + \frac{1}{r}f' = 0 \tag{5A}$$

$$g'' + \frac{1}{r}g' = h\beta^2 \tag{6A}$$

$$h'' + \frac{1}{r}h' = -g\beta^2 \tag{7A}$$

$$\ell + \frac{1}{r}\ell' = m\frac{\beta^2}{2} \tag{8A}$$

$$m'' + \frac{1}{r}m' = -\ell\frac{\beta^2}{2} \tag{9A}$$

$$\beta^2 = \frac{2\omega}{\alpha} \tag{10A}$$

Defining the following sets of functions

$$p = g - jh \qquad (11A) \qquad\qquad q = \ell - jm \qquad (12A)$$

where $j = (-1)^{1/2}$

results in the rewriting of the differential equation and its accompanying boundary conditions as follows:

$$\left.\begin{aligned} f'' + \frac{1}{r} f' &= 0 && (5A)\\ p'' + \frac{1}{r} p' &= j\beta^2 p && (13A)\\ q'' + \frac{1}{r} q' &= j\frac{\beta^2}{2} q && (14A)\end{aligned}\right\} \quad r_1 < r < r_2$$

$$\left.\begin{aligned} kNp' - 2j\omega Mp &= \frac{I_o^2 R}{2} && (15A)\\ jkNq' + \omega Mq &= -2iI_o R && (16A)\end{aligned}\right\} \quad \text{at } r = r_1$$

$$\left.\begin{aligned} p' &= 0 && (17A)\\ q' &= 0 && (18A)\end{aligned}\right\} \quad \text{at } r = r_2$$

Therefore the partial differential equation(1A) is now expressed in terms of three ordinary differential equations (5A), (13A), and (14A), which are easily solved, yielding:

$$f = \frac{-Q'(1-\gamma)}{2\pi kL} \ln r + C_o \qquad (19A)$$

$$p = AJ_o(\beta j^{3/2} r) + BK_o(\beta j^{1/2} r) \qquad (20A)$$

$$q = CJ_o\left(\frac{\beta}{\sqrt{2}} j^{3/2} r\right) + DK_o\left(\frac{\beta}{\sqrt{2}} j^{1/2} r\right) \qquad (21A)$$

C_o, A, B, C, and D are constants of integration dependent upon boundary conditions, 2A, 3A, and 15A through 18A. The constants A, B, C, and D are composed of real and imaginary parts, just as the functions p and q ,by definition,are complex variables. Thus, to separate these functions into their real and imaginary parts, the

Bessel Functions may be separated in the following manner (7, 10, 11):

$$J_0(zj^{3/2}) = \text{ber } z + j \text{ bei } z; \quad K_0(zj^{1/2}) = \text{ker } z + j \text{ kei } z$$

$$J_\nu(zj^{3/2}) = \text{ber}_\nu\, z + j \text{ bei}_\nu\, z; \quad j^{-\nu}K_\nu(zj^{1/2}) = \text{ker}_\nu\, z + j \text{ kei}_\nu\, z$$

Accordingly the functions p and q can be separated into their real and imaginary parts, as indicated in the original definition of these functions (Equations 11A and 12A) and may be resubstituted into the original trial function, Equation 4A.

Finally by assuming that the temperature fluctuations are damped out at $r = r_2$ and therefore that t_2 is constant, C_0 will equal t_2. All the necessary arbitrary constants are thus specified, and the boundary value problem is solved. An additional modification is to combine similar trigonometric functions into one function by the use of a phase angle.

APPENDIX C

Final Result of the Boundary Value Problem

$$t_1 = t_2 + \frac{(i^2 + I^2)(R(1-\gamma)}{2\pi kL} \ln \frac{r_2}{r_1} + \frac{I^2 R E_3''}{\pi r_1^2 c' \rho' L} \cdot \cos(2\omega\theta + \phi) +$$

$$\frac{2\sqrt{2}\, i I R E_2''}{\pi r_1^2 c' \rho' L} \cos(\omega\theta + \epsilon)$$

where I = effective current = $\frac{I_o}{\sqrt{2}}$

$$(E_3'')^2 = (A_1 \operatorname{ber} \beta r_1 + A_2 \operatorname{bei} \beta r_1 - A_3 \operatorname{ker} \beta r_1 - A_4 \operatorname{kei} \beta r_1)^2 +$$

$$(A_1 \operatorname{bei} \beta r_1 - A_2 \operatorname{ber} \beta r_1 - A_3 \operatorname{kei} \beta r_1 + A_4 \operatorname{ker} \beta r_1)^2$$

$$(E_2'')^2 = \left[B_1 \operatorname{ber} \frac{\beta r_1}{\sqrt{2}} + B_2 \operatorname{bei} \frac{\beta r_1}{\sqrt{2}} - B_3 \operatorname{ker} \frac{\beta r_1}{\sqrt{2}} - B_4 \operatorname{kei} \frac{\beta r_1}{\sqrt{2}}\right]^2 +$$

$$\left[B_1 \operatorname{bei} \frac{\beta r_1}{\sqrt{2}} - B_2 \operatorname{ber} \frac{\beta r_1}{\sqrt{2}} - B_3 \operatorname{kei} \frac{\beta r_1}{\sqrt{2}} + B_4 \operatorname{ker} \frac{\beta r_1}{\sqrt{2}}\right]^2$$

$$\tan \phi = \frac{A_1 \operatorname{bei}\beta r_1 - A_2 \operatorname{ber}\beta r_1 - A_3 \operatorname{kei}\beta r_1 + A_4 \operatorname{ker}\beta r_1}{A_1 \operatorname{ber}\beta r_1 + A_2 \operatorname{bei}\beta r_1 - A_3 \operatorname{ker}\beta r_1 - A_4 \operatorname{kei}\beta r_1}$$

$$\tan \epsilon = \frac{B_1 \operatorname{bei}\frac{\beta r_1}{\sqrt{2}} - B_2 \operatorname{ber}\frac{\beta r_1}{\sqrt{2}} - B_3 \operatorname{kei}\frac{\beta r_1}{\sqrt{2}} + B_4 \operatorname{ker}\frac{\beta r_1}{\sqrt{2}}}{B_1 \operatorname{ber}\frac{\beta r_1}{\sqrt{2}} + B_2 \operatorname{bei}\frac{\beta r_1}{\sqrt{2}} - B_3 \operatorname{ker}\frac{\beta r_1}{\sqrt{2}} + B_4 \operatorname{kei}\frac{\beta r_1}{\sqrt{2}}}$$

$$A_1 = \frac{\operatorname{kei}_1 \beta r_2 \left[S_1 + \frac{c\rho}{c'\rho'} \frac{\sqrt{2}}{\beta r_1} S_2\right] - \operatorname{ker}_1 \beta r_2 \left[S_3 + \frac{c\rho}{c'\rho'} \frac{\sqrt{2}}{\beta r_1} S_4\right]}{2(S_1^2 + S_3^2) + \frac{4\sqrt{2}}{\beta r_1} \frac{c\rho}{c'\rho'} (S_2 S_1 + S_3 S_4) + \left(\frac{2c\rho}{\beta r_1 c'\rho'}\right)^2 (S_2^2 + S_4^2)} =$$

$$\frac{a \operatorname{kei}_1 \beta r_2 - b \operatorname{ker}_1 \beta r_2}{\Gamma}$$

$$A_2 = \frac{a\ \mathrm{ker}_1\beta r_2 + b\ \mathrm{kei}_1\beta r_2}{\Gamma} \qquad A_3 = \frac{a\ \mathrm{bei}_1\beta r_2 - b\ \mathrm{ber}_1\beta r_2}{\Gamma}$$

$$A_4 = \frac{a\ \mathrm{ber}_1\beta r_2 + b\ \mathrm{bei}_1\beta r_2}{\Gamma}$$

$$B_1 = \frac{\mathrm{kei}_1 \frac{\beta r_2}{\sqrt{2}}\left[S_5 + \frac{2}{\beta r_1}\ \frac{c\rho}{c'\rho'}\ S_6\right] - \mathrm{ker}_1 \frac{\beta r_2}{\sqrt{2}}\left[S_7 + \frac{3}{\beta r_1}\ \frac{c\rho}{c'\rho'} S_8\right]}{(S_5{}^2+S_7{}^2) + \frac{4}{\beta r_1}\ \frac{c\rho}{c'\rho'}(S_5S_6+S_7S_8) + \left(\frac{2c\rho}{\beta r_1 c'\rho'}\right)^2 (S_6{}^2+S_8{}^2)} =$$

$$\frac{c\ \mathrm{kei}_1 \frac{\beta r_2}{2} - d\ \mathrm{ker}_1 \frac{\beta r_2}{\sqrt{2}}}{\Lambda}$$

$$B_2 = \frac{c\ \mathrm{ker}_1 \frac{\beta r_2}{\sqrt{2}} + d\ \mathrm{kei}_1 \frac{\beta r_2}{\sqrt{2}}}{\Lambda} \qquad B_3 = \frac{c\ \mathrm{bei}_1 \frac{\beta r_2}{\sqrt{2}} - d\ \mathrm{ber}_1 \frac{\beta r_2}{\sqrt{2}}}{\Lambda}$$

$$B_4 = \frac{c\ \mathrm{ber}_1 \frac{\beta r_2}{\sqrt{2}} + d\ \mathrm{bei}_1 \frac{\beta r_2}{\sqrt{2}}}{\Lambda}$$

$$S_1 = \begin{cases} -\mathrm{ker}_1\beta r_2\,\mathrm{ber}\beta r_1 \\ +\mathrm{kei}_1\beta r_2\,\mathrm{bei}\beta r_1 \\ +\mathrm{ber}_1\beta r_2\,\mathrm{ker}\beta r_1 \\ -\mathrm{bei}_1\beta r_2\,\mathrm{kei}\beta r_1 \end{cases} \qquad S_2 = \begin{cases} +\mathrm{ker}_1\beta r_2(\mathrm{bei}_1\beta r_1 - \mathrm{ber}_1\beta r_1) \\ +\mathrm{kei}_1\beta r_2(\mathrm{bei}_1\beta r_1 + \mathrm{ber}_1\beta r_1) \\ +\mathrm{ber}_1\beta r_2(\mathrm{ker}_1\beta r_1 - \mathrm{ker}_1\beta r_1) \\ -\mathrm{bei}_1\beta r_2(\mathrm{ker}_1\beta r_1 + \mathrm{ker}_1\beta r_1) \end{cases}$$

$$S_3 = \begin{cases} -\mathrm{ker}_1\beta r_2\,\mathrm{bei}\beta r_1 \\ -\mathrm{kei}_1\beta r_2\,\mathrm{ber}\beta r_1 \\ +\mathrm{ber}_1\beta r_2\,\mathrm{kei}\beta r_1 \\ +\mathrm{bei}_1\beta r_2\,\mathrm{ker}\beta r_2 \end{cases} \qquad S_4 = \begin{cases} -\mathrm{ker}_1\beta r_2(\mathrm{bei}_1\beta r_1 + \mathrm{ber}_1\beta r_1) \\ +\mathrm{kei}_1\beta r_2(\mathrm{bei}_1\beta r_1 - \mathrm{ber}_1\beta r_1) \\ +\mathrm{ber}_1\beta r_2(\mathrm{ker}_1\beta r_1 + \mathrm{kei}_1\beta r_1) \\ +\mathrm{bei}_1\beta r_2(\mathrm{ker}_1\beta r_1 - \mathrm{kei}_1\beta r_1) \end{cases}$$

$$S_5 = \left\{ \begin{array}{l} -\mathrm{ker}_1 \frac{\beta r_2}{\sqrt{2}} \ \mathrm{bei} \ \frac{\beta r_1}{\sqrt{2}} \\ -\mathrm{kei}_1 \frac{\beta r_2}{\sqrt{2}} \ \mathrm{ber} \ \frac{\beta r_1}{\sqrt{2}} \\ -\mathrm{ber}_1 \frac{\beta r_2}{\sqrt{2}} \ \mathrm{kei} \ \frac{\beta r_1}{\sqrt{2}} \\ +\mathrm{bei}_1 \frac{\beta r_2}{\sqrt{2}} \ \mathrm{ker} \ \frac{\beta r_1}{\sqrt{2}} \end{array} \right.$$

$$S_6 = \left\{ \begin{array}{l} -\mathrm{ker}_1 \frac{\beta r_2}{\sqrt{2}} \left(\mathrm{bei}_1 \frac{\beta r_1}{\sqrt{2}} + \mathrm{ber}_1 \frac{\beta r_1}{\sqrt{2}}\right) \\ +\mathrm{kei}_1 \frac{\beta r_2}{\sqrt{2}} \left(\mathrm{bei}_1 \frac{\beta r_1}{\sqrt{2}} - \mathrm{ber}_1 \frac{\beta r_1}{\sqrt{2}}\right) \\ +\mathrm{ber}_1 \frac{\beta r_2}{\sqrt{2}} \left(\mathrm{ker}_1 \frac{\beta r_1}{\sqrt{2}} + \mathrm{kei}_1 \frac{\beta r_1}{\sqrt{2}}\right) \\ +\mathrm{bei}_1 \frac{\beta r_2}{\sqrt{2}} \left(\mathrm{ker}_1 \frac{\beta r_1}{\sqrt{2}} - \mathrm{kei}_1 \frac{\beta r_1}{\sqrt{2}}\right) \end{array} \right.$$

$$S_7 = \left\{ \begin{array}{l} \mathrm{ker}_1 \frac{\beta r_2}{\sqrt{2}} \mathrm{bei} \ \frac{\beta r_1}{\sqrt{2}} \\ -\mathrm{kei}_1 \frac{\beta r_2}{\sqrt{2}} \ \mathrm{ber} \frac{\beta r_1}{\sqrt{2}} \\ -\mathrm{ber}_1 \frac{\beta r_2}{\sqrt{2}} \ \mathrm{kei} \ \frac{\beta r_1}{\sqrt{2}} \\ -\mathrm{bei}_1 \frac{\beta r_2}{\sqrt{2}} \ \mathrm{ker} \ \frac{\beta r_1}{\sqrt{2}} \end{array} \right.$$

$$S_8 = \left\{ \begin{array}{l} -\mathrm{ker}_1 \frac{\beta r_2}{\sqrt{2}} \left(\mathrm{bei}_1 \frac{\beta r_1}{\sqrt{2}} - \mathrm{ber}_1 \frac{\beta r_1}{\sqrt{2}}\right) \\ -\mathrm{kei}_1 \frac{\beta r_2}{\sqrt{2}} \left(\mathrm{bei}_1 \frac{\beta r_1}{\sqrt{2}} + \mathrm{ber}_1 \frac{\beta r_1}{\sqrt{2}}\right) \\ -\mathrm{ber}_1 \frac{\beta r_2}{\sqrt{2}} \left(\mathrm{ker}_1 \frac{\beta r_1}{\sqrt{2}} - \mathrm{kei}_1 \frac{\beta r_1}{\sqrt{2}}\right) \\ +\mathrm{bei}_1 \frac{\beta r_2}{\sqrt{2}} \left(\mathrm{ker}_1 \frac{\beta r_1}{\sqrt{2}} + \mathrm{kei}_1 \frac{\beta r_1}{\sqrt{2}}\right) \end{array} \right.$$

APPENDIX D

Summary of Principle Methods of Measuring Thermal Conductivity of Fluids

The following principles have been used for the measurement of thermal conductivity of fluids:

A. Steady State Methods

1. An electrically heated wire in a much larger cooled concentric cylinder filled with the gas or liquid (2, 8).

2. An electrically heated cylinder in a slightly larger concentric cylinder filled with the gas or liquid (5).

3. A disk of the liquid between one fluid-cooled plate and one fluid-heated (3) or electrically heated (1) plate.

4. Two concentric cylinders, one heated and one cooled, of which at least one is non-radiant and non-absorbent (4).

B. Unsteady State Methods

1. A constant electrical current is passed through a thin wire in a cylinder of the liquid initially at uniform temperature. The change of temperature with time of a point close to the wire is measured (12).

2. A tube containing the gas is electrically heated by a sudden flow of electricity. The pressure of the gas is recorded as a function of time (6).

3. A tube is heated by an alternating current of very low frequency, and the time is measured for the temperature wave to reach the center of the viscous liquid within the tube (13).

The steady state methods are mathematically very much simpler than the unsteady state ones. Methods A-1 and A-2 are the most popular methods. They are suitable for gases because the heat losses in parallel to the main conduction through the gas are concentrated at the ends,

can be minimized by using long cylinders, and can be corrected for by blank runs. Usually they are not considered suitable for high temperatures because of the radiation correction, which is expected to be high as well as to vary with time due to change in surface condition. Recently, however, a silver cell has been employed with a constant radiation correction of only 6% at 775°C (9). Method A-3 is not suitable for gases because of the relatively greater magnitude of the losses.

Method A-4 constituted an initial effort to develop a technique for high temperature gaseous thermal conductivity determination which is free of the wall-to-wall radiation correction. As in method A-2, two concentric cylinders are employed with the test gas between. However the inner cylinder is constructed of a non-absorbent solid, and cooled (or heated) by a high velocity stream of non-absorbent gas. Thus no net flow of heat by wall-to-wall radiation exists. In actual practice a thin fused quartz inner tube was employed and an effective absorptivity of 14% was obtained instead of a value near zero obtainable with the more fragile KCl, NaCl, or CaF_2 tubes. Average gas temperatures up to about 500°C were reached. At higher temperatures the effective absorptivity would have been even lower. End effects were determined empirically by calibration runs with air at low temperatures.

The unsteady methods are in general all quite recent, having been published subsequent to the initiation of the present study. Method B-1 is not suitable for gases because the thermocouple would influence the gas temperature due to its high heat capacity. Method B-2 was shown (6) not to be reliable because of its dependence on an unknown average of C_p and C_v. Method B-3 is only suitable for highly viscous liquids, having been specifically developed for molten glass. Furthermore, the values obtained are uncorrected for radiation effects.

Method A-4 seems to be the most suitable for the measurement of the thermal conductivity of gases at yet higher temperatures of all the possible methods that have been employed previously. The purpose of the present study is to investigate another possible method which will measure thermal conductivity without interference by wall-to-wall radiation. It is planned to push forward the application of whichever method seems more suitable.

NOMENCLATURE

E	Voltage drop across wire
E_3	Third harmonic component of voltage
E_2	Second harmonic component of voltage
E_r	Heat radiated per unit surface area of wire
F_A	Geometric Radiation Factor
F_E	Non-black factor of surfaces
I_o	Maximum value of alternating current
I	Effective value of alternating current
L	Length of wire
M	Heat capacity of the wire $\pi r_1^2 c' \rho' L$
N	Surface area of wire
Q'	Average power input to wire
R	Resistance, electrical, of wire
$\bar{R}$	Resistance of wire at its average temperature, $\bar{t}_1$
R_o	Resistance of wire extrapolated linearly to 0°C = $R/(1 + st)$
T_1	Absolute temperature of wire
T_2	Absolute temperate of wall
c	Heat capacity of gas
c'	Heat capacity of wire
c_p	Heat capacity at constant pressure
c_v	Heat capacity at constant volume
h	Heat transfer coefficient for radiation
i	Direct current in wire
j	$\sqrt{-1}$
k	Thermal conductivity of gas
q_c	Heat conducted through gas
q_r	Heat radiated from wire to wall
r	Radial distance from center of wire
r_1	Radius of wire
r_2	Radius of outer jacket
s	Change in resistance with temperature/R_o
t	Temperature
t_1	Temperature of wire at any time
t_2	Temperature of outer jacket

$\bar{t}_1$	Average wire temperature
α	Thermal diffusivity of the fluid
β	$\sqrt{\frac{2\omega}{\alpha}}$
γ	Fraction of power input dissipated by radiation
Δ	Phase lag of third harmonic in wire voltage behind the fundamental or the current
ϵ	Phase angle for the fundamental of the temperature fluctuation with respect to the current
ϕ	Phase angle for second harmonic of temperature fluctuation
θ	Time
ρ	Density of the fluid
ρ'	Density of wire
σ	Stefan-Boltzman Constant
ω	Angular velocity = $2\pi\cdot$(frequency)

Dimensionless Groups

$$E_3'' = \frac{E_3 \sqrt{2}\pi r_1^2 c'\rho' L\omega}{I^3 \bar{R}^2}\left(\frac{1}{s} + \bar{t}\right)$$

$$E_2'' = \frac{E_2 \pi r_1^2 c'\rho' L\omega}{2 i I^2 \bar{R}^2}\left(\frac{1}{s} + \bar{t}\right)$$

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